APPLICATION OF THE POINT TRANSFORMATION METHOD TO QUASI-PERIODIC OSCILLATIONS OF NONLINEAR SYSTEMS

(O PRIMENENII METODA TOCHECHNYKH PREOBRAZOVANII K KVAZIPERIODICHRSKIM KOLEBANIIAM NELINEINYKH SISTEM)

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V. Kh. KHARASAKHAL
(Alma-Ata)

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The following system of differential equations is examined

$$dx_s / dt = f_s(t, x_1, \ldots, x_m) \qquad (s = 1, \ldots, m) \tag{1}$$

Here $f_s(t, x_1, \ldots, x_m)$ are quasi-periodic functions with respect to t, with periods $\omega_1, \ldots, \omega_n$. Consequently f_s will be diagonal [1] functions of periodic functions $\Phi_s(u_1, \ldots, u_n, x_1, \ldots, x_m)$ with periods ω_k with respect to variables u_k, i.e., $f_s(t, x_1, \ldots, x_m) = \Phi_s(t, \ldots, t, x_1, \ldots, x_m)$.

The following system is examined along with system (1)

$$\frac{\partial x_s}{\partial u_1} + \frac{\partial x_s}{\partial u_2} + \ldots + \frac{\partial x_s}{\partial u_n} = \Phi_s(u_1, \ldots, u_n, x_1, \ldots, x_m) \qquad (s = 1, \ldots, m) \tag{2}$$

The question of the existence and uniqueness of solutions of system (2) was examined in [2]. In the same place it was shown that the periodic solution of system (2) generates quasi-periodic solution of system (1) on the diagonal $u_1 = u_2 = \ldots = u_n = t$. In addition to these quasi-periodic solutions system (1) may also have quasi-periodic solutions which are generated by nonperiodic solutions of system (2). Such solutions are not examined.

Through the use of an analysis, which was proposed by N.P. Erugin for ordinary equations [3], to system (2) it is possible to establish, utilizing diagonal $u_k = t$, that if system (1) has a quasi-periodic solution $\varphi_k(t)$ with the frequency base γ, then either the functions f_s will be quasi-periodic with respect to t with a frequency base commensurable with γ, or they will become quasi-periodic with frequency base γ after substitution of x_k by φ_k; in this case the functions f_s generally speaking also may not be quasi-periodic, or they will be quasi-periodic with a frequency base β, not commensurable with the frequency base γ.

Definition 1. Let $(x_1^\circ, \ldots, x_m^\circ)$ be a fixed point. We will say that the system of functions $\Phi_s(u_1, \ldots, u_n, x_1, \ldots, x_m)$ $(s = 1, \ldots, m)$ depends in a definite manner on the variables $u_1, \ldots, u_n$ at the point $(x_1^\circ, \ldots, x_m^\circ)$, if just one of the functions

$h_s(u_1, \ldots, u_n) = \Phi_s(u_1, \ldots, u_n, x_1^\circ, \ldots, x_m^\circ)$ is not constant [4].

The sum total of the functions $\Phi_s(u_1, \ldots, u_n, x_1, \ldots, x_m)$ $(s = 1, \ldots, m)$ depends on the variables $u_1, \ldots, u_n$ in a definite manner if it depends in a definite manner on these variables at any point $(x_1^\circ, \ldots, x_n^\circ)$.

Let Q denote the set of those points $(x_1^\circ, \ldots, x_m^\circ)$, in which the system of functions $\Phi_s(u_1, \ldots, u_n, x_1, \ldots, x_m)$ does not depend on the variables u_k in a definite manner. By the method proposed in reference [4] it is possible to prove the following: let the functions Φ_s of system (2) be periodic with respect to the variables u_k with respective periods ω_k. Let $x_s = \psi_s(u_1, \ldots, u_n)$ be a periodic solution of system (2) with periods δ_k and let the quantities δ_k/ω_k be irrational. Then $(\psi_1(u_1, \ldots, u_n), \ldots, \psi_m(u_1, \ldots, u_n)) \in Q$ for any point $(u_1^\circ, \ldots, u_n^\circ)$ (in particular for points $(u_1^\circ, \ldots, u_n^\circ)$, which are located on the diagonal).

Corollary 1. If the system of functions $\Phi_s(u_1, \ldots, u_n, x_1, \ldots, x_m)$ depends on the variables $u_1, \ldots, u_n$ in a determinate manner and if $x_s = \psi_s(u_1, \ldots, u_n)$ is a periodic solution of system (2) with periods $\delta_1, \ldots, \delta_n$, then the quantities δ_k/ω_k are rational.

Corollary 2. If the functions $f_s(t)$ in system (1) depend in a determinate manner on the variable t, then the frequency bases of functions f_s and of functions φ_s, which are solutions of system (1), are commensurable.

In the following we will consider that the functions $\Phi_s(u_1, \ldots, u_n, x_1, \ldots, x_m)$ depend in a determinate manner on the variables $u_1, \ldots, u_n$.

Let E_n be the Euclidean space of quantities $u_1, \ldots, u_n$. Vector z with components $x_1 = x_1(u_1, \ldots, u_n), \ldots, x_m = x_m(u_1, \ldots, u_n)$, where $x_k(u_1, \ldots, u_n)$ are real continuous functions, will be called a point m-dimensional metric space N. The metric of this space is defined by the equation

$$\rho(z_1, z_2) = \sup\left(\sum_{i=1}^{m}(x_{i1} - x_{i2})^2\right)^{1/2}$$

Here x_{i1} and x_{i2} are components of the vectors z_1 and z_2 respectively. We note that any solution of equations (2) is a point of space N. By means of the equations

$$y_k(u_1, \ldots, u_n) = F_k(x_1(u_1, \ldots, u_n), \ldots, x_m(u_1, \ldots, u_n)) \qquad (k = 1, \ldots, m) \quad (3)$$

let each point $P(x_1, \ldots, x_m)$ of the space N be uniquely transformed into another point of this space $P_1(y_1, \ldots, y_m)$. In this case we will say that equations (3) define the point transformation T of the space N into itself [5].

$$P_1 = TP \quad (4)$$

Point P_2 is obtained from point P by means of a double transformation T^2, if $P_2 = TP_1 = T(TP)$. An analogous transformation, consisting of k-fold successive application of the transformation T, is denoted by T^k.

Definition 2. Point P^* will be called a fixed point of the transformation T if the transformation T transfers point P^* into itself, i.e.,

$$TP^* = P^* \text{ or } \quad x_k^*(u_1, \ldots, u_n) = F_k(x_1^*(u_1, \ldots, u_n), \ldots, x_m^*(u_1, \ldots, u_n)) \quad (k = 1, \ldots, m)$$

The sum total of the points $P(x_1, \ldots, x_m)$, for which $\rho(P, P^*) < \varepsilon$ will be called the ε - neighborhood of point $P^*(x_1^*, \ldots, x_m^*)$

The fixed point P^* will be called asymptotically stable on a small scale if for any point P belonging to a sufficiently small ε-neighborhood of P^* the condition $\rho(T^k P, P^*) < \varepsilon_k$ is satisfied, where $\varepsilon_k \to 0$ as $k \to \infty$ and $\max \varepsilon_k \to 0$ as $\varepsilon \to 0$.

The fixed point P^* is called unstable if for some $\varepsilon > 0$ some, however small, neighborhood of P^* there are points P which, under successive application of the transformation T, exceed the boundaries of the ε-neighborhood of the fixed point P^*.

The solution $Z(u_1, \ldots, u_n) = \{x_1(u_1, \ldots, u_n), \ldots, x_m(u_1, \ldots, u_n)\}$ of system (2), which satisfies the initial condition

$$z_{01}(u_1^\circ, u_2, \ldots, u_n) = \{x_1(u_1^\circ, u_2, \ldots, u_n), \ldots, x_m(u_1^\circ, u_2, \ldots, u_n\})$$

will be denoted by $z(u_1, \ldots, u_n, u_1^\circ, z_{01})$.

The vector function $z(u_1, \ldots, u_n, u_1^\circ, z_{01})$ will be continuous with respect to all variables and its basic properties are expressed by the following equations

$$z(u_1^\circ, u_2, \ldots, u_n, u_1^\circ, z_{01}) \equiv z_{01} \tag{5}$$

$$z(u_1^{(2)}, u_2, \ldots, u_n, u_1^{(1)}, z(u_1^{(1)}, u_2, \ldots, u_n, u_1^\circ, z_{01})) = z(u_1^{(2)}, u_2, \ldots, u_n, u_1^\circ, z_{01})$$

It is assumed that each solution of system (2) is defined with respect to u_1 in the interval $[0, \omega_1]$. Then the equation

$$T(z_{01}) = z(\omega_1, u_2, \ldots, u_n, 0, z_{01})$$

will be called the Poincaŕe - Andronov operator T_1 for transfomation of the hyperplane $u_1 = 0$ into itself [6]. Solution $z(u_1, \ldots, u_n, 0, z_{01})$ of system (2) is periodic with respect to u_1 with period ω_1 and satisfies the equation

$$z(\omega_1, u_2, \ldots, u_n, 0, z_{01}) = z_{01}$$

i.e., the initial condition which determines the periodic solution, is the fixed point of the Poincaré - Andronov operator. Conversely, let z_{01} be the fixed point of the operator T_1. Then from equation (5) it follows that

$$z(u_1 + \omega_1, u_2, \ldots, u_n, 0, z_{01}) = z(u_1 + \omega_1, u_2, \ldots, u_n, \omega_1, z(\omega_1, u_2, \ldots, u_n, 0, z_{01})) =$$
$$= z(u_1 + \omega_1, u_2, \ldots, u_n, \omega_1, z_{01}) \tag{6}$$

However, from the periodicity of the right-hand sides of system (2) with respect to u_1 it follows that

$$z(u_1 + \omega_1, u_2, \ldots, u_n, \omega_1, z_{01}) = z(u_1, u_2, \ldots, u_n, 0, z_{01})$$

Therefore it follows from (6) that $z(u_1, \ldots, u_n, 0, z_{01})$ is a periodic solution of system (2), with respect to u_1. In a similar manner it is possible to examine the Poincaré-Andronov operators $T_2, \ldots, T_n$.

Consequently, for system (2) to have a periodic solution $z_j(u_1, \ldots, u_n, u_j^\circ, z_{0j})$ with respect to the variable u_j with period ω_j, it is necessary and sufficient for the operator T_j to have fixed points. We write

$$z_j(u_1^0, u_2, u_1, \ldots, u_n u_1, u_j^\circ, z_{0j}) = \varphi_j(u_2, \ldots, u_1)$$

Let the operators $T_1, T_2, \ldots, T_n$ have fixed points and let $\varphi_j = \varphi$ $(j = 1, \ldots, n)$.

Then periodic solutions z_j with period ω_j with respect to the variable u_j coalesce (by virtue of uniqueness) into one solution z of system (2) periodic with respect to all variables $u_1, \ldots, u_n$ with periods $\omega_1, \ldots, \omega_n$ respectively. By the same token the following theorem is valid.

Theorem. For system (1) to have a quasi-periodic solution generated by a periodic solution of system (2) it is necessary and sufficient for the operators $T_1, \ldots, T_n$ to have fixed points and for $\varphi_j = \varphi$ $(j = 1, \ldots, n)$.

Let $\sigma > 0$ be a selected small value. We denote by ν_σ the set of those points of space E_n which are located in the σ-neighborhood of the diagonal $u_1 = u_2 = \ldots = u_n$. Let $M(u_1, \ldots, u_n)$ be some point from the set ν_σ. If any one coordinate $u_j \to \infty$, then for point M to remain within ν_σ, it is necessary that all other coordinates $u_k \to \infty$.

Definition. The solution $z(u_1, \ldots, u_n) = \{x_1(u_1, \ldots, u_n), \ldots, x_m(u_1, \ldots, u_n)\}$ of system (2) with initial conditions $\overline{x_i}(u_1^\circ, u_2, \ldots, u_n)$ will be referred to as asymptotically stable in the sense of Liapunov if for any $\epsilon > 0$ given in advance, such an $r > 0$ can be found that for any other solution $y(u_1, \ldots, u_n)$ of system (2) with initial conditions

$$y_i(u_1^\circ, u_2, \ldots, u_n) = x_i(u_1^\circ, u_2, \ldots, u_n) + \delta_i(u_2, \ldots, u_n)$$

where $\|\delta\{\delta_1, \ldots, \delta_m\}\| < r$ (norm in the sense of metric in N) the following is applicable for all finite values $u_1, \ldots, u_n$ from the set ν_σ

$$\|y(u_1, \ldots, u_n) - z(u_1, \ldots, u_n)\| < \varepsilon$$

and simultaneously

$$\|y(u_1, \ldots, u_n) - z(u_1, \ldots, u_n)\| \to 0 \qquad \text{for } u_j \to 0$$

with the condition that $(u_1, \ldots, u_n) \in v_\sigma$.

It is possible to show that in the region ν_σ there is correspondence not only between periodic solutions and fixed points of the transformation T_j, but also correspondence between their stabilities. From Liapunov's stability of the periodic solution of system (2) the stability of the fixed points follows, and conversely (in so far as the periods of solution are multiples of the periods of system (2)), from the stability of the fixed points of the transformations Liapunov's stability of the corresponding periodic solution follows.

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